The `Meet` **and** `Join`**, Constraints Between Them, and Their Transformation under Outermorphism**

A supplementary discussion by Greg Grunberg of Sections 5.1–5.7 of the textbook *Geometric Algebra for Computer Science*, by Dorst, Fontijne, & Mann (2007)

Given two subspaces $\mathcal{A}$ and $\mathcal{B}$ of the overall vector space, the largest subspace common to both of them is called the `meet` of those subspaces, and as a set is the intersection $\mathcal{A} \cap \mathcal{B}$ of those subspaces. The `join` of the two given subspaces is the smallest superspace common to both of them, and as a set is the sum $\mathcal{A} + \mathcal{B} = \{\mathbf{x}_1 + \mathbf{x}_2 : \mathbf{x}_1 \in \mathcal{A} \text{ and } \mathbf{x}_2 \in \mathcal{B}\}$ of those subspaces. $\mathcal{A} + \mathcal{B}$ is not usually a "direct sum" $\mathcal{A} \bigoplus \mathcal{B}$, as the decomposition $\mathbf{x}_1 + \mathbf{x}_2$ of a nonzero element of the `join` is uniquely determined only when $\mathcal{A} \cap \mathcal{B} = \{0\}$. The system of subspaces, with its subset partial ordering and `meet` and `join` operations, is an example of the type of algebraic system called a "lattice".

Recall that a subspace $\mathcal{A}$ can be represented by a blade $\mathbf{A}$, with $\mathcal{A} = \{\mathbf{x} : \mathbf{x} \wedge \mathbf{A} = 0\}$. If $\mathcal{A}$ is unoriented, then any scalar multiple $\alpha\mathbf{A}$ $(\alpha \neq 0)$ also represents $\mathcal{A}$; if $\mathcal{A}$ is oriented, then the scalar must be positive. Our emphasis is on the algebra of the representing blades, so hereafter we usually do not refer to $\mathcal{A}$ and $\mathcal{B}$ themselves but rather to their representatives $\mathbf{A}$ and $\mathbf{B}$. When necessary we will abuse language and refer to the "subspace" $\mathbf{A}$ or $\mathbf{B}$ rather than to $\mathcal{A}$ or $\mathcal{B}$. We will examine the `meet` and `join` of two given blades $\mathbf{A}$ and $\mathbf{B}$, both before and after application of an (invertible) transformation f to get new blades $\bar{\mathbf{A}} = f[\mathbf{A}]$ and $\bar{\mathbf{B}} = f[\mathbf{B}]$. We first define `meet` and `join` of blades and examine the consequences of those definitions. Then we will see how to obtain a `meet` and `join` for the image blades $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$.

A `meet` $\mathbf{M} = \mathbf{A} \cap \mathbf{B}$ is defined to be any blade $\mathbf{M}$ which satisfies the condition

$$\boxed{\forall \mathbf{x} : [\mathbf{x} \wedge \mathbf{M} = 0] \Leftrightarrow [\mathbf{x} \wedge \mathbf{A} = 0 \text{ and } \mathbf{x} \wedge \mathbf{B} = 0]\,,}$$

while a `join` $\mathbf{J} = \mathbf{A} \cup \mathbf{B}$ is defined to be any blade which satisfies

$$\boxed{\forall \mathbf{x} : [\mathbf{x} \wedge \mathbf{J} = 0] \Leftrightarrow [\exists \mathbf{x_1} \exists \mathbf{x_2} : \mathbf{x_1} \wedge \mathbf{A} = 0 \text{ and } \mathbf{x_2} \wedge \mathbf{B} = 0 \text{ and } \mathbf{x_1} + \mathbf{x_2} = \mathbf{x}]\,.}$$

The `meet` condition says that the subspace (determined by) $\mathbf{M}$ is the largest subspace of both the subspace (determined by) $\mathbf{A}$ and the subspace (determined by) $\mathbf{B}$, while the `join` condition says that $\mathbf{J}$ is the smallest superspace of both $\mathbf{A}$ and $\mathbf{B}$.

Notation borrowed or adapted from other areas of mathematics can suggest things that aren't true, but not borrowing would proliferate notation without limit and incur the wrath of the typesetter. We therefore make some cautionary remarks so as to lend a perspective to the symbols used here. The "$\cup$″ symbol for the `join` operation should be viewed as suggesting an operation in some sense dual to the `meet` operation "$\cap$". Our $\cup$ and $\cap$ are suggestive of the notation "$\vee$" and "$\wedge$" used in lattice theory for its join and meet operations. Because of our preemptive use of "$\wedge$" for the outer product, we cannot simply appropriate lattice theory's notation, but $\cup$ and $\cap$ seem in most ways to be a good substitute. But "$\cup$" and "$\cap$" are also used in set theory. It turns out that the $\mathbf{A} \cap \mathbf{B}$ subspace is indeed the set intersection of the subspaces $\mathbf{A}$ and $\mathbf{B}$, but some thought will show that to be a happy accident, for the definition of the $\mathbf{A} \cap \mathbf{B}$ subspace involves not just set theory operations but also vector space concepts. The definition of the $\mathbf{A} \cup \mathbf{B}$ subspace, being dependent on the concept of vector addition, also involves more than just set operations, but for $\mathbf{A} \cup \mathbf{B}$ there is no happy accident: $\mathbf{A} \cup \mathbf{B}$ it is *not* the set union of the subspaces $\mathbf{A}$ and $\mathbf{B}$, but is rather their sum (not *direct* sum). A final danger we mention is that, because of the subspace/blade correspondence, one might be tempted to think that the subspace `join` $\mathcal{A}+\mathcal{B}$ has blade correspondent $\mathbf{A}+\mathbf{B}$. Avoid that temptation! If nothing else, the `join` $\mathbf{J}$ must be a blade, but in $\mathbb{R}^n$, $n>3$, addition of blades doesn't necessarily result in another blade.

The two conditions which define `meet` and `join` determine the attitudes of the blades $\mathbf{M}$ and $\mathbf{J}$ but not their magnitudes or orientations, as can be seen from the facts that $\lambda\mathbf{M}$ satisfies the `meet` condition iff M does, and $\mu\mathbf{J}$ satisfies the `join` condition iff $\mathbf{J}$ does. We do not preclude λ or μ from being negative, which would reverse the orientation of $\mathbf{M}$ or $\mathbf{J}$. We can speak of *a* `meet` or *a* `join`, but without further specification we cannot speak of *the* `meet` or *the* `join`. As far as the definitions go, there is no coupling of the magnitude and orientation of $\mathbf{M}$ with those of $\mathbf{J}$. Instead a coupling is introduced by adoption of a *constraint* between $\mathbf{M}$ and $\mathbf{J}$. The constraint we choose to use takes three equivalent forms,

$$\boxed{\mathbf{J}=\left(\mathbf{A}\lfloor\mathbf{M}^{-1}\right)\wedge\mathbf{M}\wedge\left(\mathbf{M}^{-1}\rfloor\mathbf{B}\right)} \quad \text{or} \quad \boxed{\mathbf{M}=\left(\mathbf{B}\rfloor\mathbf{J}^{-1}\right)\rfloor\mathbf{A}} \quad \text{or} \quad \boxed{\left(\mathbf{M}\rfloor\mathbf{J}^{-1}\right)=\left(\mathbf{B}\rfloor\mathbf{J}^{-1}\right)\wedge\left(\mathbf{A}\rfloor\mathbf{J}^{-1}\right).}$$

At this point in the discussion the constraint in its various forms is not particularly meaningful, but we can at least see that the constraint serves to determine the magnitude and orientation of one of the blades $\mathbf{M}$ and $\mathbf{J}$ given those same attributes for the other blade: If we make the change $\mathbf{M}\longrightarrow\lambda\mathbf{M}$, the first version of the constraint requires $\mathbf{J}\longrightarrow\lambda^{-1}\mathbf{J}$; the change $\mathbf{J}\longrightarrow\mu\mathbf{J}$ and the second version of the constraint requires $\mathbf{M}\longrightarrow\mu^{-1}\mathbf{M}$. ($\lambda\mu=1$, of course.) Once the constraint is imposed and the magnitude and orientation of one of the blades is chosen, we may speak of *the* `meet` $\mathbf{M}$ and *the* `join` $\mathbf{J}$.

Let us try to motivate the contraint's first version. Given $\mathbf{M}$, we can find its orthogonal complements in $\mathbf{A}$ and $\mathbf{B}$ by using the left and right contractions to remove $\mathbf{M}$ from those blades in an orthogonal manner,

$$\boxed{\mathbf{A}':=\mathbf{A}\lfloor\mathbf{M}^{-1}} \quad \text{and} \quad \boxed{\mathbf{B}':=\mathbf{M}^{-1}\rfloor\mathbf{B}.}$$

Use of the right contraction to form $\mathbf{A}'$ enables us to recover $\mathbf{A}$ by right outer multiplication with $\mathbf{M}$, while use of the left contraction to form $\mathbf{B}'$ enables recovery of $\mathbf{B}$ by left outer multiplication with $\mathbf{M}$:

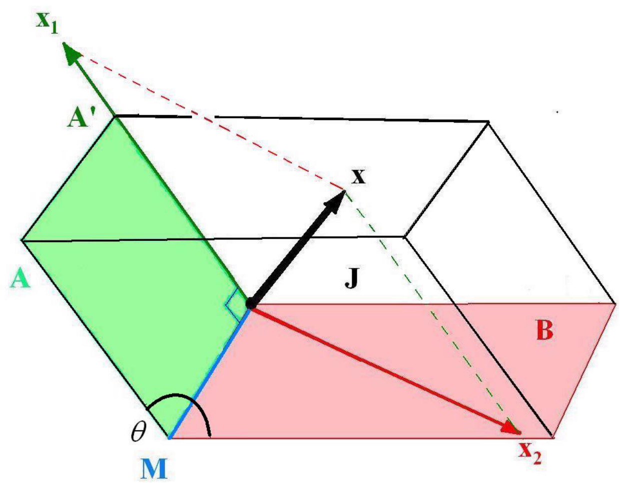
$$\boxed{\mathbf{A}'\wedge\mathbf{M}=\left(\mathbf{A}\lfloor\mathbf{M}^{-1}\right)\wedge\mathbf{M}=\mathbf{A}} \quad \text{and} \quad \boxed{\mathbf{M}\wedge\mathbf{B}'=\mathbf{M}\wedge\left(\mathbf{M}^{-1}\rfloor\mathbf{B}\right)=\mathbf{B}.}$$

This is convenient, for then we can write either $\mathbf{A}'\wedge\mathbf{M}\wedge\mathbf{B}'=(\mathbf{A}'\wedge\mathbf{M})\wedge\mathbf{B}'=\mathbf{A}\wedge\mathbf{B}'$ or $\mathbf{A}'\wedge\mathbf{M}\wedge\mathbf{B}'=\mathbf{A}'\wedge(\mathbf{M}\wedge\mathbf{B}')=\mathbf{A}'\wedge\mathbf{B}$ without worrying about whether the order of the factors $\mathbf{A}'$, $\mathbf{M}$, $\mathbf{B}'$ must be changed in order to combine $\mathbf{A}'$ with $\mathbf{M}$ or to combine $\mathbf{M}$ with $\mathbf{B}'$. With the orthogonal complements in our possession, it's reasonable to try creating $\mathbf{J}$ by putting together $\mathbf{A}'$ (the part of $\mathbf{A}$ not $\mathbf{M}$), $\mathbf{M}$, and $\mathbf{B}'$ (the part of $\mathbf{B}$ not $\mathbf{M}$). We can write this variously as

$$\begin{aligned}
\mathbf{J} &= \mathbf{A}'\wedge\mathbf{M}\wedge\mathbf{B}' &&= \left(\mathbf{A}\lfloor\mathbf{M}^{-1}\right)\wedge\mathbf{M}\wedge\left(\mathbf{M}^{-1}\rfloor\mathbf{B}\right) && \text{(combine } \mathbf{A} \text{ not } \mathbf{M} \text{ with } \mathbf{M} \text{ and } \mathbf{B} \text{ not } \mathbf{M}\text{)}\\
&= (\mathbf{A}'\wedge\mathbf{M})\wedge\mathbf{B}' &&= \mathbf{A}\wedge\left(\mathbf{M}^{-1}\rfloor\mathbf{B}\right) && \text{(combine } \mathbf{A} \text{ with } \mathbf{B} \text{ not } \mathbf{M}\text{)}\\
&= \mathbf{A}'\wedge(\mathbf{M}\wedge\mathbf{B}') &&= \left(\mathbf{A}\lfloor\mathbf{M}^{-1}\right)\wedge\mathbf{B} && \text{(combine } \mathbf{A} \text{ not } \mathbf{M} \text{ with } \mathbf{B}\text{),}
\end{aligned}$$

depending on what's convenient for the purpose at hand. Note that the constraint $\mathbf{J}=\mathbf{A}'\wedge\mathbf{M}\wedge\mathbf{B}'$ is a *convention* we choose to adopt. It is not required by or a consequence of the definitions of `meet` and `join`, although it must integrate with those definitions. $\mathbf{J}=\mathbf{A}'\wedge\mathbf{M}\wedge\mathbf{B}'$ is a *reasonable* convention to make. In principle one could also use some weird scalar multiple like $\frac{7}{3}\left(\mathbf{A}'\wedge\mathbf{M}\wedge\mathbf{B}'\right)=\mathbf{A}\wedge\left(\frac{7}{3}\mathbf{B}'\right)=\left(\frac{7}{3}\mathbf{A}'\right)\wedge\mathbf{B}$, which would also be a `join` if $\mathbf{A}'\wedge\mathbf{M}\wedge\mathbf{B}'$ is, but such a multiple would be needlessly complicated and not so useful for adoption as a constraint. Note that our constraint is not universally adhered to by other authors, so when consulting other texts be sure to check which conventions they use; most deviations from our convention by other texts can be expressed as a sign factor dependent on the grades of the blades involved.

Loosely speaking, taking $\mathbf{B}$ out of $\mathbf{J}$ leaves $\mathbf{A}'$ (sort of, but not precisely), which when taken out of $\mathbf{A}$ in turn leaves $\mathbf{M}$; therefore the second version $\mathbf{M} = \left(\mathbf{B}\rfloor\mathbf{J}^{-1}\right)\rfloor\mathbf{A}$ of the constraint is also reasonable. As will be seen below, the constraint's third version is closely related to the second version. Of course $\mathbf{A}' \wedge \mathbf{M} \wedge \mathbf{B}'$ had better actually be *a* `join` if the constraint declaring it to be $\mathbf{J}$ is to make any sense! Observe that $\mathbf{A}' \wedge \mathbf{M} \wedge \mathbf{B}'$ satisfies the $\Rightarrow$ portion of the `join` definition. For suppose $\mathbf{x} \wedge (\mathbf{A}' \wedge \mathbf{M} \wedge \mathbf{B}') = 0$, i.e. suppose $\mathbf{x} \wedge (\mathbf{A}' \wedge \mathbf{B}) = 0$. Within the $\mathbf{A}' \wedge \mathbf{B}$ space, in which $\mathbf{x}$ resides, obtain $\mathbf{x}_1$ and $\mathbf{x}_2$ respectively by parallel projection of $\mathbf{x}$ parallelly to $\mathbf{B}$ (*not* orthogonal projection) and to $\mathbf{A}'$ to get $\mathbf{x}_1$ and $\mathbf{x}_2$ (see the figure). Then $\mathbf{x}_1 \wedge \mathbf{A}' = 0$, whence $\mathbf{x}_1 \wedge \mathbf{A} = \mathbf{x}_1 \wedge (\mathbf{A}' \wedge \mathbf{M}) = 0$, and $\mathbf{x}_2 \wedge \mathbf{B} = 0$. By the construction of $\mathbf{x}_1$ and $\mathbf{x}_2$, the $\mathbf{x} \in \mathbf{A}' \wedge \mathbf{M} \wedge \mathbf{B}'$ with which we started is $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$. We have decomposed the subspace $\mathbf{A}' \wedge \mathbf{M} \wedge \mathbf{B}'$ into the direct sum of the (disjoint but not-necessarily-orthogonal) subspaces $\mathbf{A}'$ and $\mathbf{B}$.



Conversely, $\mathbf{A}' \wedge \mathbf{M} \wedge \mathbf{B}'$ satisfies the $\Leftarrow$ portion of the definition: If $\mathbf{x}_1 \wedge \mathbf{A} = 0$ and $\mathbf{x}_2 \wedge \mathbf{B} = 0$, then $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ satisfies

$$\mathbf{x} \wedge \mathbf{J} = \mathbf{x}_1 \wedge ((\mathbf{A}' \wedge \mathbf{M}) \wedge \mathbf{B}') + \mathbf{x}_2 \wedge (\mathbf{A}' \wedge (\mathbf{M} \wedge \mathbf{B}')) = \mathbf{x}_1 \wedge \mathbf{A} \wedge \mathbf{B}' + \mathbf{x}_2 \wedge \mathbf{A}' \wedge \mathbf{B} = 0 + 0 = 0.$$

The first version of the constraint, $\mathbf{J} = \left(\mathbf{A}\lfloor\mathbf{M}^{-1}\right) \wedge \mathbf{B}$, specifically gives $\mathbf{J}$ in terms of $\mathbf{M}$. We invert this so as to give $\mathbf{M}$ in terms in $\mathbf{J}$:

$$\begin{aligned}
\mathbf{M} &= \mathbf{M} \wedge \left(\mathbf{J}\rfloor\mathbf{J}^{-1}\right) && \text{because } \mathbf{J}\rfloor\mathbf{J}^{-1} = 1 \\
&= \mathbf{M} \wedge \left(\left(\left(\mathbf{A}\lfloor\mathbf{M}^{-1}\right) \wedge \mathbf{B}\right)\rfloor\mathbf{J}^{-1}\right) && \text{because } \mathbf{J} = \left(\mathbf{A}\lfloor\mathbf{M}^{-1}\right) \wedge \mathbf{B} \\
&= \mathbf{M} \wedge \left(\left(\mathbf{A}\lfloor\mathbf{M}^{-1}\right)\rfloor\left(\mathbf{B}\rfloor\mathbf{J}^{-1}\right)\right) && \text{because } (\mathbf{X} \wedge \mathbf{Y})\rfloor\mathbf{Z} = \mathbf{X}\rfloor(\mathbf{Y}\rfloor\mathbf{Z}) \\
&= \mathbf{M} \wedge \left(\left(\mathbf{A}\lfloor\mathbf{M}^{-1}\right) * \left(\mathbf{B}\rfloor\mathbf{J}^{-1}\right)\right) && \text{because } \left(\mathbf{A}\lfloor\mathbf{M}^{-1}\right) \text{ and } \left(\mathbf{B}\rfloor\mathbf{J}^{-1}\right) \text{ have equal grades} \\
&= \mathbf{M} \wedge \left(\mathbf{A} * \left(\mathbf{M}^{-1} \wedge \left(\mathbf{B}\rfloor\mathbf{J}^{-1}\right)\right)\right) && \text{because } (\mathbf{Z}\lfloor\mathbf{Y}) * \mathbf{X} = \mathbf{Z}\lfloor(\mathbf{Y} \wedge \mathbf{X}) \\
&= \mathbf{M} \wedge \left(\left(\mathbf{M}^{-1} \wedge \left(\mathbf{B}\rfloor\mathbf{J}^{-1}\right)\right) * \mathbf{A}\right) && \text{because } * \text{ is commutative} \\
&= \mathbf{M} \wedge \left(\left(\mathbf{M}^{-1} \wedge \left(\mathbf{B}\rfloor\mathbf{J}^{-1}\right)\right)\rfloor\mathbf{A}\right) && \text{because } \left(\mathbf{M}^{-1} \wedge \left(\mathbf{B}\rfloor\mathbf{J}^{-1}\right)\right) \text{ and } \mathbf{A} \text{ have equal grades} \\
&= \left(\mathbf{M}\rfloor\left(\mathbf{M}^{-1} \wedge \left(\mathbf{B}\rfloor\mathbf{J}^{-1}\right)\right)\right)\rfloor\mathbf{A} && \text{because } \mathbf{M} \subseteq \mathbf{A} \\
&= \left(\left(\mathbf{M}\rfloor\mathbf{M}^{-1}\right) \wedge \left(\mathbf{B}\rfloor\mathbf{J}^{-1}\right)\right)\rfloor\mathbf{A} && \text{because } \mathbf{M} \perp \left(\mathbf{B}\rfloor\mathbf{J}^{-1}\right) \\
&= \left(\mathbf{B}\rfloor\mathbf{J}^{-1}\right)\rfloor\mathbf{A} && \text{because } \mathbf{M}\rfloor\mathbf{M}^{-1} = 1.
\end{aligned}$$

In short, we have obtained the equivalent version $\mathbf{M} = \left(\mathbf{B}\rfloor\mathbf{J}^{-1}\right)\rfloor\mathbf{A}$ of the constraint. Incidentally, a similar calculation in terms of the right contraction shows $\mathbf{M} = \mathbf{B}\lfloor\left(\mathbf{J}^{-1}\lfloor\mathbf{A}\right)$.

Now $\mathbf{M} \subseteq \mathbf{A}$, so contraction into $\mathbf{J}^{-1}$ of both sides of the constraint's second version gives still another version

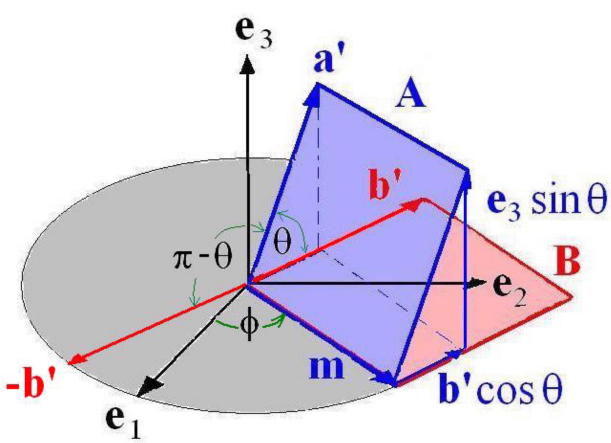
$$\mathbf{M}\rfloor\mathbf{J}^{-1} = \mathbf{M}\rfloor\left(\left(\mathbf{B}\rfloor\mathbf{J}^{-1}\right)\rfloor\mathbf{A}\right) = \left(\mathbf{B}\rfloor\mathbf{J}^{-1}\right)\wedge\left(\mathbf{A}\rfloor\mathbf{J}^{-1}\right),$$

which may be written

$$\boxed{\mathbf{M}^\bigstar = \mathbf{B}^\bigstar \wedge \mathbf{A}^\bigstar;} \qquad \text{whence} \qquad \boxed{\mathbf{M} = \left(\mathbf{B}^\bigstar \wedge \mathbf{A}^\bigstar\right)^{-\bigstar}.}$$

Here $\bigstar$ has been used to indicate *dualization with respect to the* `join` (as opposed to $*$ to indicate dualization with respect to the overall space's unit pseudoscalar $\mathbf{I}$). The last boxed equation is the geometric algebra justification for cross multiplication of the normals to two planes to find their line of intersection.

Let's use the second constraint to find the `meet` of two 2-blades in $\mathbb{R}^{3,0}$. As we want to demonstrate that our formulas give us what we think they should, we first rig the example so that we should be able to see before calculation what should be the attitudes of the `meet` and `join`. We may visualize $\mathbf{B} = \mathbf{e}_1 \wedge \mathbf{e}_2$ as a unit square with first edge $\mathbf{e}_1$ and second edge $\mathbf{e}_2$. Rotate $\mathbf{B}$'s edges by ϕ about the direction $\mathbf{e}_3$ orthogonal to the $\mathbf{B}$ plane; this rotation gives a new unit square representation $\mathbf{B} = \mathbf{m}\wedge\mathbf{b}'$ of blade $\mathbf{B}$, with orthogonal edges $\mathbf{m} = \mathbf{e}_1\cos\phi + \mathbf{e}_2\sin\phi$ and $\mathbf{b}' = -\mathbf{e}_1\sin\phi + \mathbf{e}_2\cos\phi$. Rotate the new version of $\mathbf{B}$ by θ about its first edge $\mathbf{m}$ to obtain unit square $\mathbf{A} = \mathbf{m}\wedge\mathbf{a}' = \mathbf{a}'\wedge(-\mathbf{m})$ with edges $\mathbf{m}$ and $\mathbf{a}' = \mathbf{b}'\cos\theta + \mathbf{e}_3\sin\theta = -\mathbf{e}_1\sin\phi\cos\theta + \mathbf{e}_2\cos\phi\cos\theta + \mathbf{e}_3\sin\theta$. See the figure.



The normalized 2-blade

$$\mathbf{A} = \mathbf{m}\wedge\mathbf{a}' = (\mathbf{e}_1\cos\phi + \mathbf{e}_2\sin\phi)\wedge(-\mathbf{e}_1\sin\phi\cos\theta + \mathbf{e}_2\cos\phi\cos\theta + \mathbf{e}_3\sin\theta)$$

then represents an (oriented) plane rotated θ away from the (oriented) plane of

$$\mathbf{B} = \mathbf{e}_1\wedge\mathbf{e}_2 = \mathbf{m}\wedge\mathbf{b}' = (\mathbf{e}_1\cos\phi + \mathbf{e}_2\sin\phi)\wedge(-\mathbf{e}_1\sin\phi + \mathbf{e}_2\cos\phi)$$

and meeting the $\mathbf{B}$-plane along the direction $\mathbf{m} = \mathbf{e}_1\cos\phi + \mathbf{e}_2\sin\phi$. Assuming $\theta \neq 0$ and $\theta \neq \pi$, the `join` subspace should be 3-dimensional, so we take $\mathbf{J}$ to be the 3-blade $\mathbf{J} = \mathbf{I}_3 = \mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3$. The `meet` follows from the second version of the constraint,

$$\begin{aligned}\mathbf{M} &= \left(\mathbf{B}\rfloor\mathbf{J}^{-1}\right)\rfloor\mathbf{A} = ((\mathbf{e}_1\wedge\mathbf{e}_2)\rfloor(\mathbf{e}_3\wedge\mathbf{e}_2\wedge\mathbf{e}_1))\rfloor\mathbf{A} = \mathbf{e}_3\rfloor\mathbf{A} \\ &= \mathbf{e}_3\rfloor((\mathbf{e}_1\cos\phi+\mathbf{e}_2\sin\phi)\wedge(-\mathbf{e}_1\cos\theta\sin\phi+\mathbf{e}_2\cos\theta\cos\phi+\mathbf{e}_3\sin\theta)) \\ &= -(\mathbf{e}_1\cos\phi+\mathbf{e}_2\sin\phi)\sin\theta = -\mathbf{m}\sin\theta.\end{aligned}$$

We've shown–as we should have expected from the way $\mathbf{A}$ was constructed from $\mathbf{B}$–that the `meet` $\mathbf{M}$ is codirectional with $\mathbf{m}$. An added bonus comes from noting that $\|\mathbf{M}\| = |\sin\theta|$, where θ is the angle between $\mathbf{B}$ and $\mathbf{A}$; this is true not just for our example but also for general blades $\mathbf{B}$ and $\mathbf{A}$ when $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{J}$ all have unit magnitude. Now $\|\mathbf{M}\| = |\sin\theta|$ has maximum value 1 at $\theta = \frac{\pi}{2}$, but $|\sin\theta| \longrightarrow 0$ as $\theta \longrightarrow 0$ or $\theta \longrightarrow \pi$. But $\mathbf{M}$ doesn't actually vanish at $\theta = 0$ or $\theta = \pi$. Rather the critical assumption that $\mathbf{J} = \mathbf{I}_3$ becomes invalid; we have $\mathbf{A} = \mathbf{B}$ for $\theta = 0$ and $\mathbf{A} = -\mathbf{B}$ for $\theta = \pi$, so the correct `join` when $\theta = 0$ or $\theta = \pi$ is not the trivector $\mathbf{I}_3$ but rather the bivector $\mathbf{I}_2 = \mathbf{e}_1 \wedge \mathbf{e}_2$; rather than $\mathbf{M}$ vanishing at $\theta = 0$ or $\theta = \pi$, it becomes identical with $\mathbf{I}_2$ or $-\mathbf{I}_2$. This grade- and magnitude-changing discontinuity is why $\mathbf{M}$ and $\mathbf{J}$ are described as "*mostly* linear". In a further exploration of the example, computation shows the complements of $\mathbf{M}$ in $\mathbf{A}$ and $\mathbf{B}$ to be

$$\mathbf{A}' = \mathbf{A}\lfloor\mathbf{M}^{-1} = (\mathbf{m}\wedge\mathbf{a}')\lfloor\frac{-\mathbf{m}}{\sin\theta} = \frac{\mathbf{a}'}{\sin\theta} \qquad \text{and} \qquad \mathbf{B}' = \mathbf{M}^{-1}\rfloor\mathbf{B} = \frac{\mathbf{m}}{-\sin\theta}\rfloor\mathbf{m}\wedge\mathbf{b}' = \frac{-\mathbf{b}'}{\sin\theta}.$$

These are orthogonal to $\mathbf{M}$,

$$\mathbf{A}'\cdot\mathbf{M} = \frac{\mathbf{a}'}{\sin\theta}\cdot(-\mathbf{m}\sin\theta) = -\mathbf{a}'\cdot\mathbf{m} = 0 \qquad \text{and} \qquad \mathbf{B}'\cdot\mathbf{M} = \frac{-\mathbf{b}'}{\sin\theta}\cdot(-\mathbf{m}\sin\theta) = \mathbf{b}'\cdot\mathbf{m} = 0,$$

and when combined with that blade give back the original blades $\mathbf{A}$ and $\mathbf{B}$,

$$\mathbf{A}'\wedge\mathbf{M} = \frac{\mathbf{a}'}{\sin\theta}\wedge(-\mathbf{m}\sin\theta) = \mathbf{m}\wedge\mathbf{a}' = \mathbf{A} \qquad \text{and} \qquad \mathbf{M}\wedge\mathbf{B}' = (-\mathbf{m}\sin\theta)\wedge\frac{-\mathbf{b}'}{\sin\theta} = \mathbf{m}\wedge\mathbf{b}' = \mathbf{B}.$$

The expression $\mathbf{A}'\wedge\mathbf{M}\wedge\mathbf{B}'$ gives back $\mathbf{J}$, as it should:

$$\mathbf{A}'\wedge(\mathbf{M}\wedge\mathbf{B}') = \frac{\mathbf{a}'}{\sin\theta}\wedge(\mathbf{m}\wedge\mathbf{b}') = \frac{\mathbf{a}'}{\sin\theta}\wedge\mathbf{B} = \frac{(\mathbf{b}'\cos\theta+\mathbf{e}_3\sin\theta)\wedge(\mathbf{e}_1\wedge\mathbf{e}_2)}{\sin\theta} = \frac{\mathbf{I}_3\sin\theta}{\sin\theta} = \mathbf{J}.$$

As depicted by the figure, the angle between $\mathbf{A}'$ and $\mathbf{B}'$ is $\arccos\left(\frac{\mathbf{A}'}{\|\mathbf{A}'\|}\cdot\frac{\mathbf{B}'}{\|\mathbf{B}'\|}\right) = \arccos(-\cos\theta) = \pi - \theta.\frac{\pi}{2}$. Thus it is *not* true that $\mathbf{A}' \perp \mathbf{B}'$, and so the decomposition $\mathbf{A}'\wedge\mathbf{M}\wedge\mathbf{B}'$ of $\mathbf{J}$ is not an orthogonal one.

Finally we shall apply an invertible transformation f to $\mathbf{A}$ and $\mathbf{B}$ and examine the question of the `meet` and `join` of the resultant image blades

$$\boxed{\bar{\mathbf{A}} := f[\mathbf{A}]} \quad \text{and} \quad \boxed{\bar{\mathbf{B}} := f[\mathbf{B}].}$$

Since

$$\begin{aligned}[\mathbf{y}\wedge f[\mathbf{M}] = 0] \quad &\Leftrightarrow \quad \left[f^{-1}[\mathbf{y}]\wedge\mathbf{M} = 0\right] \\ &\Leftrightarrow \quad \left[f^{-1}[\mathbf{y}]\wedge\mathbf{A} = 0 \text{ and } f^{-1}[\mathbf{y}]\wedge\mathbf{B} = 0\right] \\ &\Leftrightarrow \quad \left[\mathbf{y}\wedge f[\mathbf{A}] = 0 \text{ and } \mathbf{y}\wedge f[\mathbf{B}] = 0 \Leftrightarrow \mathbf{y}\wedge\bar{\mathbf{A}} = 0 \text{ and } \mathbf{y}\wedge\bar{\mathbf{B}} = 0\right]\end{aligned}$$

and

$$\begin{aligned}[\mathbf{y} \wedge f[\mathbf{J}] = 0] \quad &\Leftrightarrow \quad \left[f^{-1}[\mathbf{y}] \wedge \mathbf{J} = 0\right] \\ &\Leftrightarrow \quad \left[\exists \mathbf{x_1} \exists \mathbf{x_2} : \mathbf{x_1} \wedge \mathbf{A} = 0 \text{ and } \mathbf{x_2} \wedge \mathbf{B} = 0 \text{ and } \mathbf{x_1} + \mathbf{x}_2 = f^{-1}[\mathbf{y}]\right] \\ &\Leftrightarrow \quad [\exists \mathbf{x_1} \exists \mathbf{x_2} : f[\mathbf{x_1}] \wedge f[\mathbf{A}] = 0 \text{ and } f[\mathbf{x_2}] \wedge f[\mathbf{B}] = 0 \text{ and } f[\mathbf{x_1}] + f[\mathbf{x_2}] = \mathbf{y}] \\ &\Leftrightarrow \quad \left[\exists \mathbf{y}_1 \exists \mathbf{y}_2 : \mathbf{y}_1 \wedge \bar{\mathbf{A}} = 0 \text{ and } \mathbf{y}_2 \wedge \bar{\mathbf{B}} = 0 \text{ and } \mathbf{y}_1 + \mathbf{y}_2 = \mathbf{y}\right],\end{aligned}$$

we see that $f[\mathbf{M}]$ is *a* `meet` of $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$ and $f[\mathbf{J}]$ is *a* `join` of those blades.

But–as noted above–the constraint is a separate question from the `meet` and `join` definitions, so we must ask whether

$$\bar{\mathbf{M}} := f[\mathbf{M}] \quad \text{and} \quad \bar{\mathbf{J}} := f[\mathbf{J}]$$

are related by the constraint. The blade $\bar{\mathbf{J}} = f[\mathbf{J}] = f\left[\left(\mathbf{A} \lfloor \mathbf{M}^{-1}\right) \wedge \mathbf{B}\right]$ expands to

$$\bar{\mathbf{J}} = \left(f[\mathbf{A}] \lfloor \bar{f}^{-1}\left[\mathbf{M}^{-1}\right]\right) \wedge f[\mathbf{B}] = \left(\bar{\mathbf{A}} \lfloor \bar{f}^{-1}\left[\mathbf{M}^{-1}\right]\right) \wedge \bar{\mathbf{B}},$$

which expression is structurally similar to the right-hand side of the as-yet-unproven first version of the constraint

$$\bar{\mathbf{J}} = \left(\bar{\mathbf{A}} \lfloor \bar{\mathbf{M}}^{-1}\right) \wedge \bar{\mathbf{B}}.$$

This structural similarity suggests that we try to find $\bar{\mathbf{M}}$ in terms of $\bar{\mathbf{J}}$ in the same way used above to invert the first version of the constraint between $\mathbf{M}$ and $\mathbf{J}$:

$$\begin{aligned}\bar{\mathbf{M}} \quad &= \bar{\mathbf{M}} \wedge \left(\bar{\mathbf{J}} \rfloor \bar{\mathbf{J}}^{-1}\right) & &= \bar{\mathbf{M}} \wedge \left(\left(\left(\bar{\mathbf{A}} \lfloor \bar{f}^{-1}\left[\mathbf{M}^{-1}\right]\right) \wedge \bar{\mathbf{B}}\right) \rfloor \bar{\mathbf{J}}^{-1}\right) \\ &= \bar{\mathbf{M}} \wedge \left(\left(\bar{\mathbf{A}} \lfloor \bar{f}^{-1}\left[\mathbf{M}^{-1}\right]\right) \rfloor \left(\bar{\mathbf{B}} \rfloor \bar{\mathbf{J}}^{-1}\right)\right) & &= \bar{\mathbf{M}} \wedge \left(\left(\bar{\mathbf{A}} \lfloor \bar{f}^{-1}\left[\mathbf{M}^{-1}\right]\right) * \left(\bar{\mathbf{B}} \rfloor \bar{\mathbf{J}}^{-1}\right)\right) \\ &= \bar{\mathbf{M}} \wedge \left(\bar{\mathbf{A}} * \left(\bar{f}^{-1}\left[\mathbf{M}^{-1}\right] \wedge \left(\bar{\mathbf{B}} \rfloor \bar{\mathbf{J}}^{-1}\right)\right)\right) & &= \bar{\mathbf{M}} \wedge \left(\left(\bar{f}^{-1}\left[\mathbf{M}^{-1}\right] \wedge \left(\bar{\mathbf{B}} \rfloor f \bar{\mathbf{J}}^{-1}\right)\right) * \bar{\mathbf{A}}\right) \\ &= \bar{\mathbf{M}} \wedge \left(\left(\bar{f}^{-1}\left[\mathbf{M}^{-1}\right] \wedge \left(\bar{\mathbf{B}} \rfloor \bar{\mathbf{J}}^{-1}\right)\right) \rfloor \bar{\mathbf{A}}\right) & &= \left(\bar{\mathbf{M}} \rfloor \left(\bar{f}^{-1}\left[\mathbf{M}^{-1}\right] \wedge \left(\bar{\mathbf{B}} \rfloor \bar{\mathbf{J}}^{-1}\right)\right)\right) \rfloor \bar{\mathbf{A}} \\ &= \left(\left(\bar{\mathbf{M}} \rfloor \bar{f}^{-1}\left[\mathbf{M}^{-1}\right]\right) \wedge \left(\bar{\mathbf{B}} \rfloor \bar{\mathbf{J}}^{-1}\right)\right) \rfloor \bar{\mathbf{A}} & &= \left(\bar{\mathbf{M}} * \bar{f}^{-1}\left[\mathbf{M}^{-1}\right]\right) \left(\left(\bar{\mathbf{B}} \rfloor \bar{\mathbf{J}}^{-1}\right) \rfloor \bar{\mathbf{A}}\right)\end{aligned}$$

Now use the definitions of $\bar{\mathbf{M}}$ and $\bar{f}$,

$$\begin{aligned}\bar{\mathbf{M}} \quad &= \left(f[\mathbf{M}] * \bar{f}^{-1}\left[\mathbf{M}^{-1}\right]\right) \left(\left(\bar{\mathbf{B}} \rfloor \bar{\mathbf{J}}^{-1}\right) \rfloor \bar{\mathbf{A}}\right) & &= \left(\mathbf{M} * \bar{f}\left[\bar{f}^{-1}\left[\mathbf{M}^{-1}\right]\right]\right) \left(\left(\bar{\mathbf{B}} \rfloor \bar{\mathbf{J}}^{-1}\right) \rfloor \bar{\mathbf{A}}\right) \\ &= \left(\mathbf{M} * \mathbf{M}^{-1}\right) \left(\bar{\mathbf{B}} \rfloor \bar{\mathbf{J}}^{-1}\right) \rfloor \bar{\mathbf{A}} & &= \left(\bar{\mathbf{B}} \rfloor \bar{\mathbf{J}}^{-1}\right) \rfloor \bar{\mathbf{A}},\end{aligned}$$

to arrive at the second version of the constraint between $\bar{\mathbf{M}}$ and $\bar{\mathbf{J}}$.

We summarize: If we chose $\bar{\mathbf{J}} := f[\mathbf{J}]$ to be *the* `join` of $\bar{\mathbf{A}} := f[\mathbf{A}]$ and $\bar{\mathbf{B}} := f[\mathbf{B}]$, then $\bar{\mathbf{M}} := f[\mathbf{M}]$ will be *the* `meet`; and conversely. $\bar{\mathbf{M}}$ and $\bar{\mathbf{J}}$ will then satisfy any of the equivalent constraints

$$\bar{\mathbf{J}} = \left(\bar{\mathbf{A}} \lfloor \bar{\mathbf{M}}^{-1}\right) \wedge \bar{\mathbf{M}} \wedge \left(\bar{\mathbf{M}}^{-1} \rfloor \bar{\mathbf{B}}\right) \quad \text{or} \quad \bar{\mathbf{M}} = \left(\bar{\mathbf{B}} \rfloor \bar{\mathbf{J}}^{-1}\right) \rfloor \bar{\mathbf{A}} \quad \text{or} \quad \bar{\mathbf{M}} \rfloor \bar{\mathbf{J}}^{-1} = \left(\bar{\mathbf{B}} \rfloor \bar{\mathbf{J}}^{-1}\right) \wedge \left(\bar{\mathbf{A}} \rfloor \bar{\mathbf{J}}^{-1}\right).$$

Note that *dualization is taken with repect to* $\bar{\mathbf{J}} = f[\mathbf{J}]$, not with respect to $\mathbf{J}$.

If we assign a magnitude and orientation to either $\mathbf{M}$ or $\mathbf{J}$, the constraint fixes the other blade and in particular its magnitude and orientation; $\bar{\mathbf{M}}$ and $\bar{\mathbf{J}}$ are in turn fixed by f. Thus the initial assignment of magnitude and orientation attributes to one blade ripples through to the others.

1. Suppose we had reason to take a specific normalization for the preimage join, say $\|\mathbf{J}\| = 1$. This would fix $\|\mathbf{M}\|$, $\left\|\bar{\mathbf{M}}\right\| = \|f[\mathbf{M}]\|$, and $\left\|\bar{\mathbf{J}}\right\| = \|f[\mathbf{J}]\|$. So long as we insisted on both $\|\mathbf{J}\| = 1$ and $\bar{\mathbf{J}} = f[\mathbf{J}]$, we would have no freedom to require $\left\|\bar{\mathbf{J}}\right\| = 1$, . (We are not requiring the transformation f to be

orthogonal, so it may change the magnitude of a blade on which it acts.)

2. Relaxation of either requirement would allow us to make $\left\|\bar{\mathbf{J}}\right\| = 1$. If we kept the requirement $\bar{\mathbf{J}} = f[\mathbf{J}]$, we could adjust the magnitude of $\mathbf{J}$ and thereby adjust $\left\|\bar{\mathbf{J}}\right\| = \|f[\mathbf{J}]\|$. The constraint would determine $\|\mathbf{M}\|$ and $\left\|\bar{\mathbf{M}}\right\|$).

3. Alternately we could keep $\|\mathbf{J}\| = 1$ and scale $f[\mathbf{J}]$ to obtain $\bar{\mathbf{J}}$. That is to say we could take

$$\bar{\mathbf{J}} = \left(\frac{\pm 1}{\|f[\mathbf{J}]\|}\right) f[\mathbf{J}] \quad \text{and} \quad \bar{\mathbf{M}} = (\pm \|f[\mathbf{J}]\|)\, f[\mathbf{M}]$$

   instead of $\bar{\mathbf{J}} = f[\mathbf{J}]$ and $\bar{\mathbf{M}} = f[\mathbf{M}]$.

4. Suppose $\mathbf{J}$ is an eigenblade of f, $f[\mathbf{J}] = \lambda\mathbf{J}$, with nonzero eigenvalue λ. Then we could use $\mathbf{J}$ as the `join` of $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$ as well as the `join` of $\mathbf{A}$ and $\mathbf{B}$, and we would have

$$\bar{\mathbf{J}} = \mathbf{J} = \lambda^{-1} f[\mathbf{J}] \quad \text{and} \quad \bar{\mathbf{M}} = \lambda f[\mathbf{M}].$$

   This can be done only when $\mathbf{J}$ is an eigenblade of f; otherwise $\mathbf{J}$ and $f[\mathbf{J}]$ will not be co-attitudinal, so $\mathbf{J}$ will not be a `join` of $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$.

We make one final observation about $\mathbf{A}\cap\mathbf{B}$ and $\mathbf{A}\cup\mathbf{B}$. We have called $\cup$ the "`join` operation". A binary operation is usually regarded as a *function* in the sense of producing from its arguments a *unique* value, but the value blade $\mathbf{A}\cup\mathbf{B}$ is not uniquely determined from the argument blades $\mathbf{A}$ and $\mathbf{B}$. This problem goes away if we regard $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{A}\cup\mathbf{B}$ as subspaces instead of blades. Of course a similar remark holds about $\mathbf{A}\cap\mathbf{B}$. Can $\mathbf{A}\cap\mathbf{B}$ and $\mathbf{A}\cup\mathbf{B}$ be made into *blade* operations? We could impose $\|\mathbf{A}\cup\mathbf{B}\| = 1$ as a second constraint and thereby remove any indeterminancy in the magnitudes of $\mathbf{M} = \mathbf{A}\cap\mathbf{B}$ and $\mathbf{J} = \mathbf{A}\cup\mathbf{B}$. But a final indeterminancy in the orientation of one of $\mathbf{A}\cap\mathbf{B}$ and $\mathbf{A}\cup\mathbf{B}$ would still exist. The author knows no way of removing that indeterminancy.